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CONVEXLY INDEPENDENT SETS

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A family of pairwise disjoint compact convex sets is called convexly independent, if none of its members is contained in the convex hull of the union of the other members of the family. The main result of the paper gives an upper bound for the maximum cardinality $h(k,n)$ of a family $\mathcal{F}$ of mutually disjoint compact convex sets such that any subfamily of at most k members of $\mathcal{F}$ is convexly independent, but no subfamily of size n is.

By an *oval* we mean a compact convex set in the plane. Let $\mathcal{F}$ be a family of mutually disjoint ovals. A member A of $\mathcal{F}$ is called a *vertex* of $\mathcal{F}$ if

$$A \not\subset \operatorname{conv}\Big(\bigcup_{X \in \mathcal{F} \setminus \{A\}} X \Big),$$

otherwise A is called an *internal member* of $\mathcal{F}$. The family $\mathcal{F}$ is called *convexly independent*, if all of its members are vertices.

In [1] we proved that for any integer $n \geq 4$, there is a smallest natural number $g(n)$ such that if $\mathcal{F}$ is a family of more than $g(n)$ mutually disjoint ovals with the property that any three members of $\mathcal{F}$ are convexly independent, then $\mathcal{F}$ contains n convexly independent members. This result generalizes a well-known theorem of Erdős and Szekeres [3] which concerns the case when all members of $\mathcal{F}$ are points. In [1] and [2], we proved that $g(4)=4$ and $g(5)=8$; however, for arbitrary n, we were able to give only a very poor upper bound for $g(n)$. As a matter of fact, we did not give an explicit upper bound for $g(n)$, but following the main steps of the proof in [1] and using known upper bounds for the Ramsey numbers one readily obtains that

$$g(n) \leq t_n\Big(t_{n-1}\Big(\dots(t_1(c_n n))\dots\Big)\Big),$$

where $t_n(x)$ is the n-th tower function defined by $t_1(x)=x$ and

$$t_{n+1}(x) = 2^{t_n(x)}.$$

In this article we shall consider families $\mathcal{F}$ of mutually disjoint ovals with the property that, for a fixed integer $k \geq 3$, any $m \leq k$ members of $\mathcal{F}$ are convexly independent. It will be convenient to refer to this property as property H_k. Further,

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we say that $\mathcal{F}$ satisfies property H^n if no n members of $\mathcal{F}$ are convexly independent, and $\mathcal{F}$ has property H_k^n if it satisfies both H_k and H^n. With these notions our theorem mentioned above states that the cardinality of a family of mutually disjoint ovals satisfying property H_3^n is bounded. It is natural to ask: What is the maximum cardinality $h(k,n)$ of a family of mutually disjoint ovals satisfying property H_k^n for given k and n, $3 \le k \le n$? The existence of the numbers $h(k,n)$ is guaranteed by the fact that property H_k^n implies H_ℓ^n for $\ell < k$. Thus we have

$$h(k,n) \le h(3,n) = g(n).$$

Of course, one can expect that, for $k > 3$, $h(k,n)$ is considerably smaller than $h(3,n)$. Our main result is an upper bound for $h(k,n)$ which is of the magnitude c^n for $k = 4$ and of the magnitude cn^2 for $k \ge 5$.

Theorem. *We have*

$$h(4,n) \le (n-4)\binom{2n-4}{n-2} - n + 7$$

and

$$h(k,n) \le (n-3)\left[\frac{n-4}{k-4}\right] + n - 1$$

for $5 \le k \le n$.

The following example shows that

$$n - 1 + \left[\frac{n-1}{k-2}\right] \le h(k,n).$$

Let $a_1, \dots a_{n-1}$ be the vertices of a convex $n-1$-gon. For $i = 1, 2, \dots, n-1$ let r_i be the point of intersection of the diagonals $a_{i-1}a_{i+1}$ and $a_i a_{i+2}$. Here, and throughout the construction we consider all indices modulo $n-1$. Let p_i and q_i $i = 1, \dots, n-1$ be the bisecting points of the segments $a_i r_i$ and $r_i a_{i+1}$ respectively. For $\ell = n, n+1, \dots, m = n - 1 + \left[\frac{n-1}{k-2}\right]$ we define the set A_ℓ as

$$A_\ell = \operatorname{conv}\left(\{q_{\ell(k-2)-k+3}, p_{\ell(k-2)-k+4}, \dots q_{\ell(k-2)}, p_{\ell(k-2)+1}\}\right).$$

Then it is easy to check that the family $\mathcal{F}$ consisting of the ovals $A_1 = a_1$, $A_2 = a_2, \dots, A_{n-1} = a_{n-1}$, $A_n, \dots, A_m$ satisfies property H_k^n.

Generally, we can add further members to the family obtained in this way without violating property H_k^n. In Fig. 1 the case $n = 13$, $k = 4$ is depicted. Here three further ovals can be inserted, and property H_4^{13} still holds. However, using this "greedy" construction, we cannot obtain a family with property H_k^n whose cardinality exceeds cn.

Our theorem, together with Ramsay's theorem, yields a considerable improvement upon our previous upper bound for $g(n) = h(3,n)$. We recall that, for positive integers k, ℓ_1 and ℓ_2, the Ramsey number $R_k(\ell_1, \ell_2)$ is the minimal integer m with the property that if the k-element subsets of a set S of cardinality m are coloured with two colours, red and blue, say, then there is a subset S_1 of S with cardinality ℓ_1, such that all k element subsets of S_1 are red, or there is a subset S_2 of S with

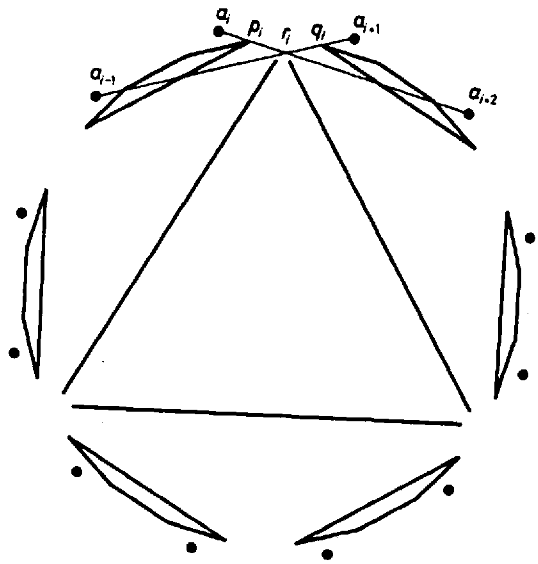

Fig. 1

cardinality ℓ_2 such that all k-element subsets of S_2 are blue [5]. Let $\mathcal{F}$ be a family of mutually disjoint ovals satisfying property H_3^n. Consider the 4-member subfamilies of $\mathcal{F}$ and colour them red or blue according as they are convexly independent or not. Observe that if $\mathcal{F}_1$ is a subfamily of $\mathcal{F}$ such that all 4-tuples of $\mathcal{F}_1$ are red, then $\mathcal{F}_1$ satisfies property H_4^n. Similarly, if $\mathcal{F}_2$ is a subfamily of $\mathcal{F}$ such that all 4-tuples of $\mathcal{F}_2$ are blue, then $\mathcal{F}_2$ has property H_3^4. Therefore the cardinality of $\mathcal{F}$ is less than $R_4(h(4,n)+1, h(3,4)+1)$. Noting that $R_k(\ell_1,\ell_2)$ is increasing in ℓ_1 and ℓ_2, our Theorem and the fact that $h(3,4)=4$ implies that

$$g(n) = h(3,n) < R_4\left((n-4)\binom{2n-4}{n-2} - n + 8, 5\right).$$

We obtain a better sense how big the improvement is by observing that the known inequalities $R_2(\ell_1,\ell_2) \le \binom{\ell_1+\ell_2-2}{\ell_1-1}$ (see [5] p.77) and $R_k(\ell_1,\ell_2) \le 2^{cR_{k-1}(\ell_1,\ell_2)}$ (see the proof of Ramsey's theorem on pp. 7–8 in [5]) imply that $R_4(m,5) \le t_3(cm)$. Since $\binom{2n-4}{n-2} = O(c^n)$ we get

$$g(n) \le t_4(cn).$$

Erdős and Szekeres [4] constructed a set of 2^{n-2} points satisfying property H_3^n. This yields

$$2^{n-2} \le g(n),$$

and it is conjectured that this bound is exact. Thus there is still room for improvement on the upper bound.

Proof of the Theorem. The proof of the Theorem follows in two steps. First we divide a family of ovals satisfying property H_k, $k \geq 4$, into subfamilies with a common transversal line, then we give an upper bound for the cardinality of a family of ovals with a common transversal satisfying property H_k^n.

Lemma 1. *Let $k \geq 4$ and let $\mathcal{F}$ be a family of mutually disjoint ovals satisfying property H_k. If $\mathcal{F}$ has m vertices, then there are $\left\lceil \frac{m-3}{k-3} \right\rceil$ straight lines such that any internal member of $\mathcal{F}$ meets one of these lines, and moreover, each of the lines intersects two vertices of $\mathcal{F}$.*

Proof. We observe that, for $m < k$, $\mathcal{F}$ is convexly independent and thus the lemma holds vacuously. Let $m \geq k$, and assume that the lemma is valid for any $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}'$ has m' vertices and $m' \leq m-1$. Let $A_1, \dots, A_m$ be the vertices of $\mathcal{F}$. Write $F = \operatorname{conv}(\bigcup_{A \in \mathcal{F}} A)$. First we consider the case that there is a vertex of $\mathcal{F}$, A_1 say such that $A_1 \cap \operatorname{bd} F$ is not connected. Let x and y be two points lying in different components of $A_1 \cap \operatorname{bd} F$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ consist of A_1 and of the members of $\mathcal{F}$ meeting the open half-plane on the right and on the left hand side of the oriented line xy, respectively. Then each internal member of $\mathcal{F}$ is an internal member of either $\mathcal{F}_1$ or $\mathcal{F}_2$.

Let $m_i, i = 1, 2$, be the number of vertices of $\mathcal{F}_i$. Then we have $m_i < m$ and $m_1 + m_2 = m + 1$. Thus the inductive hypotheses implies that there are $\left\lceil \frac{m_1-3}{k-3} \right\rceil + \left\lceil \frac{m_2-3}{k-3} \right\rceil \leq \left\lceil \frac{m-3}{k-3} \right\rceil$ lines with the properties described in the lemma.

We now suppose that $A_i \cap \operatorname{bd} F$ is connected for $i = 1, \dots, m$. We orient $\operatorname{bd} F$ so that $\operatorname{int} F$ lies to the left when travelling on $\operatorname{bd} F$ in positive direction. We assume that we meet the sets $A_i \cap \operatorname{bd} F$ on $\operatorname{bd} F$ in the cyclic order $1, 2, \dots, m, 1$. We write $A_{m+1} = A_1$. Let $\mathcal{F}'$ consist of those members of $\mathcal{F}$ which are contained in $\operatorname{conv}\left(\bigcup_{i=1}^{m-k+3} A_i\right)$. Then $\mathcal{F}'$ has $m-k-3$ vertices, namely the sets A_i, $i = 1, \dots, m-k+3$. By the inductive hypothesis there are $t = \left\lceil \frac{m-k}{k-3} \right\rceil$ straight lines $\ell_1, \dots, \ell_t$ such that any internal member of $\mathcal{F}'$ intersects one of the lines ℓ_i, $i = 1, \dots, t$, and each line ℓ_i meets two of the sets A_i, $i = 1, \dots, m-k+3$. Let v be a point of $A_1 \cap \operatorname{bd} F$ and w a point of $A_{m-k+3} \cap \operatorname{bd} F$. The segment vw divides F into two parts; let F' be the part on the right hand side of the oriented line vw, and let F'' be the other part. Then $F' \subset \operatorname{conv}\left(\bigcup_{i=1}^{m-k+3} A_i\right)$ and $F'' \subset \operatorname{conv}\left(\bigcup_{i=m-k+3}^{m+1} A_i\right)$. We observe that no internal member of $\mathcal{F}$ is contained in F'', by property H_k. Thus an internal member of $\mathcal{F}$ is either contained in F' and consequently, it is an internal member of $\mathcal{F}'$ or it intersects the line vw. Now the lines $\ell_1, \dots, \ell_t, \ell_{t+1} = vw$ satisfy the properties claimed in the lemma. ∎

Lemma 2. *If $\mathcal{F}$ satisfies H_5 and there is a straight line intersecting all members of $\mathcal{F}$, then $\mathcal{F}$ is convexly independent.*

Proof. Let $A_1, \dots, A_t$ be the members of $\mathcal{F}$ met by an oriented line ℓ in their natural order. As before, we write $F = \operatorname{conv}\left(\bigcup_{A\in\mathcal{F}} A\right)$. First we show that A_1 and A_t are vertices of $\mathcal{F}$. Suppose that A_1, say is an internal member of $\mathcal{F}$. Then there is a point $p \in \ell \cap \operatorname{bd} F$ such that

$$\ell \cap A_1 \subset \operatorname{conv}\left(\{p\} \cup \bigcup_{i=2}^{t} (\ell \cap A_i)\right).$$

As ℓ meets A_1 first, it follows that $p \notin \operatorname{conv}\left(\bigcup_{i=2}^{t}(\ell \cap A_i)\right)$. Thus there is a supporting line s of F, through p, which supports some A_i and A_j where $2 \le i \le j \le \ell$. Let $x \in s \cap A_i$, $y \in s \cap A_j$, $v \in \ell \cap A_j$ and $w \in \ell \cap A_i$. Then, on the one hand, the closed simple quadrilateral $xyvw$ encloses A_1, on the other hand, it is contained in $\operatorname{conv}(A_i \cup A_j)$. This is a contradiction by property H_5. Thus A_1, and similarly A_t is a vertex of $\mathcal{F}$.

Now suppose that for some i, $2 \le i \le t-1$, A_i is an internal member of $\mathcal{F}$. We know already that A_1, and A_t are vertices of $\mathcal{F}$. Let p be a point of $A_1 \cap \operatorname{bd} F$ and q a point of $A_t \cap \operatorname{bd} F$. Let $C[p,q]$ and $C[q,p]$ be the two closed arcs of $\operatorname{bd} F$ with the endpoints p and q. Both of these arcs are composed of arcs belonging to the boundary of some vertex of $\mathcal{F}$ and of straight segments belonging to lines which support two vertices of $\mathcal{F}$. Since A_1 and A_t meet $C[p,q]$ and $1 < i < t$, there exist two vertices A_h and A_j of $\mathcal{F}$ with $h < i < j$ and a line s which supports A_h, A_j and F such that $s \cap \operatorname{bd} F \subset C[p,q]$. Similarly, there are two vertices $A_{h'}$ and $A_{j'}$, of $\mathcal{F}$ with $h' < i < j'$ and a line s' which supports $A_{h'}, A_{j'}$ and F such that $s' \cap \operatorname{bd} F \subset C[q,p]$. Choosing points $x \in s \cap A_h$, $y \in s \cap A_j$, $v \in \ell \cap A_j$, $w \in \ell \cap A_h$, $x' \in s' \cap A_{h'}$, $y' \in s' \cap A_{j'}$ $v' \in \ell \cap A_{j'}$ and $w' \in \ell \cap A_{h'}$ we see that the simple closed polygon $xyvv'y'x'w'w$ contains A_i and is contained in $\operatorname{conv}(A_h \cup A_j \cup A_{h'} \cup A_{y'})$. This contradicts property H_5. Thus all members of $\mathcal{F}$ are vertices. ∎

We observe that in Lemma 2 the assumption that $\mathcal{F}$ satisfies H_5 cannot be weakened. Figure 2 shows a family $\mathcal{F}$ consisting of five straight line segments with a common transversal such that $\mathcal{F}$ satisfies property H_4, but it is not convexly independent. However, an idea of Erdős and Szekeres enables us to give an upper bound of exponential magnitude for the size of a family of mutually disjoint ovals satisfying property H_3 and sharing a common transversal.

Lemma 3. *If $\mathcal{F}$ satisfies property H_3^n and there is a straight line intersecting all members of $\mathcal{F}$, then $\mathcal{F}$ has at most $\binom{2n-4}{n-2}$ members.*

Proof. We use the notations of the previous Lemma. We recall that a supporting half-plane of a set X is bounded by a supporting line of X and contains X. For two distinct members A_i and A_j of $\mathcal{F}$ we set

$$S_{ij} = \operatorname{conv}(A_i \cup A_j),$$

and observe that there are two distinct supporting half-planes of S_{ij} which also support A_i and A_j. We orient the lines bounding these supporting half-planes so

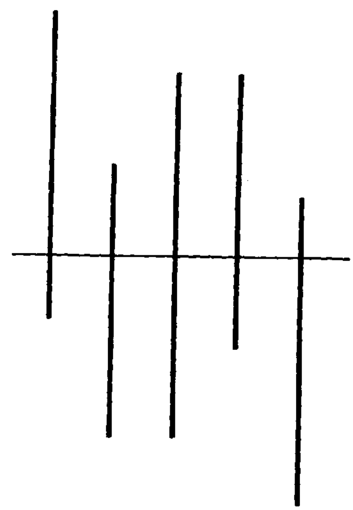

Fig. 2

that the interior of the respective half-plane lies to the left when travelling on the supporting line in positive direction. We denote the two directed lines obtained in this way by s_{ij} and s_{ji}. We choose the notations so that travelling on s_{ij} in positive direction we meet first A_i, and then A_j.

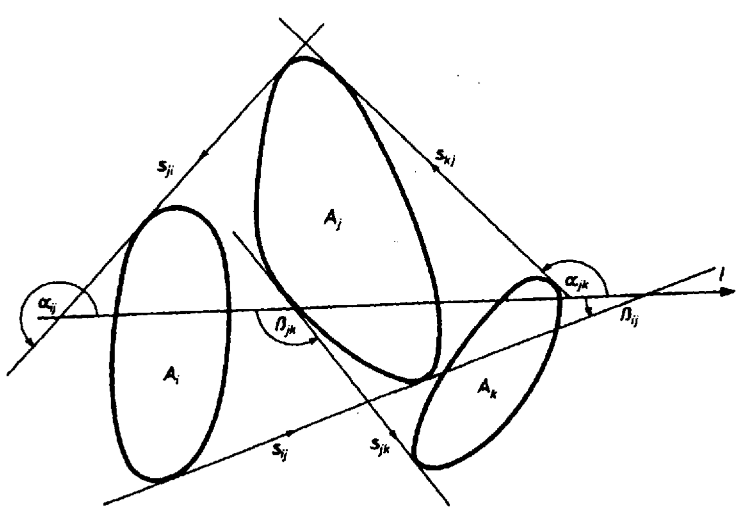

Fig. 3

We assign a positive direction of rotation in the counter-clockwise sense. For $i < j$ let α_{ij} be the angle of rotation between 0 and 2π carrying the direction of the line ℓ into the direction of s_{ji}. Let β_{ij} be the angle of rotation between 0 and 2π carrying the direction opposite to the direction of ℓ into the direction of s_{ij} (Fig. 3). We imagine that the plane in which we work is in a vertical position so that ℓ is horizontal and its direction points from left to right. Then the following notions are justified: A subfamily $\mathcal{F}'$ of $\mathcal{F}$ is (i) *strictly convex from above,* (ii) *concave from above,* (iii) *strictly convex from below* and (iv) *concave from below* if we have (i) $\alpha_{ij} > \alpha_{jk}$, (ii) $\alpha_{ij} \leq \alpha_{jk}$, (iii) $\beta_{ij} < \beta_{jk}$ and (iv) $\beta_{ij} \geq \beta_{jk}$, respectively, for any three members A_i, A_j and A_k of $\mathcal{F}'$ with increasing indices i, j and k. We observe that if $\mathcal{F}'$ is strictly convex from above or from below, then $\mathcal{F}'$ is convexly independent. Further, property H_3 implies that if $\mathcal{F}'$ is concave from one side, then it is strictly convex from the other side.

Now the argument of Erdős and Szekeres in [3] pp. 469–470 yields that, if $\mathcal{F}$ has more than $\binom{2n-4}{n-2}$ members, then $\mathcal{F}$ has a subfamily of size n which is either strictly convex or concave from above. Thus a family of more than $\binom{2n-4}{n-2}$ members cannot satisfy property H_3^n. This completes the proof of Lemma 3. ∎

Using Lemmas 1 to 3, the proof of the Theorem is very easy. Let $\mathcal{F}$ be a family of mutually disjoint ovals satisfying property $H_k^n (k \geq 4)$. Let $m \leq n-1$ be the number of vertices of $\mathcal{F}$. By Lemma 1, there are $\left\lceil \frac{m-3}{k-3} \right\rceil$ lines such that each internal member of $\mathcal{F}$ meets one of these lines. Since, furthermore, any of these lines meets two vertices of $\mathcal{F}$, the number of internal members of $\mathcal{F}$ meeting one of the given lines is at most $n-3$ for $k \geq 5$, and it is at most $\binom{2n-4}{n-2} - 2$ for $k = 4$. Thus the total number of the members of $\mathcal{F}$ is at most $m + (n-3) \left\lceil \frac{m-3}{k-3} \right\rceil \leq n - 1 + (n-3) \left\lceil \frac{n-4}{k-3} \right\rceil$ if $\mathcal{F}$ satisfies H_k^n for $k \geq 5$, and at most $m + (m-3) \left(\binom{2n-4}{n-2} - 2 \right) \leq (n-4) \binom{2n-4}{n-2} - n + 7$ if $\mathcal{F}$ satisfies H_4^n. ∎

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